Available online at www.sciencedirect.com

SCIENCE DIRECT•

ELSEVIER

Computers and Mathematics with Applications 52 (2006) 225–234

An International Journal
**computers & mathematics with applications**

www.elsevier.com/locate/camwa

# Advancing Equitability in Multiobjective Programming

D. Baatar
Department of Mathematics and Statistics
The University of Melbourne
Parkville, VIC 3010, Australia
d.baatar@ms.unimelb.edu.au

M. M. Wiecek
Department of Mathematical Sciences
Clemson University
Clemson, SC, U.S.A.
wmalgor@clemson.edu

**Abstract**—In multiobjective programming, the concept of equitable efficiency strengthens the concept of Pareto efficiency by additionally requiring that the objective functions be anonymous and satisfy the principle of transfers. The preference relation satisfying these assumptions is not related to a cone as is the Pareto preference. A complete preference structure of equitability is derived and an approach to generating equitably efficient solutions is proposed. 

**Keywords**—Equitability, Pareto efficiency, Multiobjective programming, Preference structure.

## 1. INTRODUCTION

Traditionally, multiobjective programming (MOP) and multicriteria decision making (MCDM) have been based on the concept of Pareto efficiency (optimality). Yu [1] develops a convex-cone theory for modeling preferences in MOP and gives foundations for many successful research initiatives in MOP and MCDM. MOP has also been studied in the context of other principles such as optimality based on the lexicographic ordering, max-ordering, and lexicographic max-ordering. Recent surveys on the state-of-the-art in MOP and MCDM the reader can find in [2].

Some researchers undertake efforts to generalize the convex-cone approach of Yu. Bergstresser *et al.* [3] use a convex set rather than a convex cone to represent preferences. Takeda and Nishida [4] introduce fuzzy domination structures for MOP while Hazen and Morin [5] study optimality conditions for MOP with a nonconical order. More recently, Weidner [6,7] studies scalarization approaches to multiobjective programs with preferences modeled by parameter-depending sets while Chen and Yang [8] relate a variable domination structure to a nonlinear scalarization for MOP.

Motivated by the interest in equity issues, Kostreva and Ogryczak [9] introduce the concept of equitability into MOP. While Pareto efficiency assumes that the criteria are uncomparable,

This research was supported by the Deutsche Forschungsgemeinschaft Grant Ka 477/24-1.

0898-1221/06/$ - see front matter  Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX
doi:10.1016/j.camwa.2006.08.014

equitability is based on the assumption that the criteria are not only comparable (measured on a common scale) but also anonymous (impartial). The latter makes the distribution of outcomes among the criteria more important than the assignment of outcomes to specific criteria, and therefore models equitable allocation of resources. Kostreva and Ogryczak [9] and Kostreva *et al.* [10] develop scalarization approaches to generating equitable efficient solutions of linear and nonlinear multiobjective programs. Ogryczak demonstrates equitability on portfolio optimization [11] and location problems [12].

The intention is this paper is to study equitability within the framework developed by Yu. It is shown that the preference relation representing equitability is not derived from a unique cone as are other preferences (e.g., Pareto, lexicographic) but from a finite number of cones. In the next section, the concepts of interest are defined and important results from the literature reviewed. Preliminary results are presented in Section 3 while the description of the equitability preference structure is developed in Section 4. An approach to generating equitably efficient solutions is proposed in Section 5 and Section 6 concludes the paper.

## 2. PROBLEM STATEMENT

Throughout this article the following notation is used. Let $\mathbb{R}^m$ be the Euclidean vector space and $y^1, y^2 \in \mathbb{R}^m$. $y^1 < y^2$ denotes $y_i^1 < y_i^2$ for all $i = 1, \dots, m$. $y^1 \leqq y^2$ denotes $y_i^1 \leq y_i^2$ for all $i = 1, \dots, m$. $y^1 \leq y^2$ denotes $y^1 \leqq y^2$ but $y^1 \neq y^2$.

DEFINITION 1. *Let $y^1, y^2 \in \mathbb{R}^m$ and let $\succeq$ be a relation of weak preference defined on $\mathbb{R}^m \times \mathbb{R}^m$. The corresponding relations of strict preference $\succ$ and indifference $\sim$ are defined as follows:*

1. $y^1 \succ y^2 \Leftrightarrow (y^1 \succeq y^2$ *and not* $y^2 \succeq y^1)$.
2. $y^1 \sim y^2 \Leftrightarrow (y^1 \succeq y^2$ *and* $y^2 \succeq y^1)$.

DEFINITION 2. *Preference relations satisfying the following axioms are called equitable rational preference relations:*

1. *Reflexivity: for all* $y \in \mathbb{R}^m$: $y \succeq y$.
2. *Transitivity: for all* $y^1, y^2, y^3 \in \mathbb{R}^m$: $y^1 \succeq y^2$ *and* $y^2 \succeq y^3 \Rightarrow y^1 \succeq y^3$.
3. *Strict monotonicity: for all* $y \in \mathbb{R}^m$: $y - \varepsilon e_i \succ y$ *for* $\varepsilon > 0$ *where* $e_i$ *denotes the* $i^{th}$ *unit vector in* $\mathbb{R}^m$.
4. *Impartiality: for all* $y \in \mathbb{R}^m$: $(y_1, y_2, \dots, y_m) \cong (y_{\tau(1)}, y_{\tau(2)}, \dots, y_{\tau(m)})$ *for any permutation* $\tau$ *of components of* $y$.
5. *Principle of transfers: for all* $y \in \mathbb{R}^m$: $y_i > y_j \Rightarrow y - \varepsilon e_i + \varepsilon e_j \succ y$ *for* $0 < \varepsilon < y_i - y_j$.

DEFINITION 3. *Let $y^1$, $y^2$ be in $\mathbb{R}^m$. We say that $y^1$ is equitably preferred to (equitably indifferent to, equitably dominated by) $y^2, y^1 \succ_e (\sim_e, \prec_e) y^2$ if $y^1 \succ (\sim, \prec) y^2$ for all equitable rational preference relations $\succeq$.*

Following [9], we define the ordering map $\Theta$, the cumulative ordering map $\bar{\Theta}$, and review their properties.

DEFINITION 4. *Let $y$ be in $\mathbb{R}^m$.*

1. *Let* $\Theta : \mathbb{R}^m \to \mathbb{R}^m$ *be the ordering map defined as* $\Theta(y) = (\theta_1(y), \theta_2(y), \dots, \theta_m(y))$, *where* $\theta_1(y) \geq \theta_2(y) \geq \cdots \geq \theta_m(y)$, $\theta_i(y) = y_{\tau(i)}$ *for* $i = 1, 2, \dots, m$, *and* $\tau$ *is a permutation of the set* $\{1, \dots, m\}$.
2. *Let* $\bar{\Theta} : \mathbb{R}^m \to \mathbb{R}^m$ *be the cumulative ordering map defined as* $\bar{\Theta}(y) = (\bar{\theta}_1(y), \bar{\theta}_2(y), \dots, \bar{\theta}_m(y))$, *where* $\bar{\theta}_i(y) := \sum_{j=1}^{i} \theta_j(y)$ *for* $i = 1, 2, \dots, m$.

PROPOSITION 1. *Let $y^1$ and $y^2$ be in $\mathbb{R}^m$. Then*

1. $\Theta(y^1) = \Theta(y^2)$ *iff* $\bar{\Theta}(y^1) = \bar{\Theta}(y^2)$,
2. $\Theta(y^1) \leq \Theta(y^2) \Rightarrow \bar{\Theta}(y^1) \leq \bar{\Theta}(y^2)$,
3. $y^1 \succeq_e y^2$ *iff* $\bar{\Theta}(y^1) \leqq \bar{\Theta}(y^2)$.

We refer to the space $\mathbb{R}^m$ as the outcome space. Following Yu [1], we partition this space into disjoint subsets of outcomes that are dominated by, preferred than, indifferent to, or undefined with respect to a given outcome $\bar{y} \in \mathbb{R}^m$.

DEFINITION 5. *Let $\bar{y} \in \mathbb{R}^m$. The set of points*

1. *equitably dominated by $\bar{y}$ is defined as $D(\bar{y}) := \{y \in \mathbb{R}^m : y \prec_e \bar{y}\}$;*
2. *equitably preferred to $\bar{y}$ is defined as $P(\bar{y}) := \{y \in \mathbb{R}^m : y \succ_e \bar{y}\}$;*
3. *equitably indifferent to $\bar{y}$ is defined as $I(\bar{y}) := \{y \in \mathbb{R}^m : y \sim_e \bar{y}\}$;*
4. *equitably undefined with respect to $\bar{y}$ is defined as $U(\bar{y}) := \mathbb{R}^m \setminus \{D(\bar{y}) \cup P(\bar{y}) \cup I(\bar{y})\}$.*

The triple of sets $(D(\bar{y}), P(\bar{y}), I(\bar{y}))$ is referred to as the equitability preference structure.

## 3. PRELIMINARY RESULTS

In this section we develop a matrix-based description of the ordering map $\Theta$ and the cumulative ordering map $\bar{\Theta}$ and construct related cones.

Let $I_k$, $k = 1, \dots, m!$, denote the matrices obtained by permuting columns of the $m \times m$ identity matrix

$$I_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

REMARK 1. The following properties hold for a matrix $I_k$, $k \in \{1, \dots, m!\}$:

1. $I_k I_k = I_k^\top$,
2. $I_k^\top I_k = I_k I_k^\top = I_1$.

REMARK 2. Let $y \in \mathbb{R}^m$ and $\Theta(y)$ be the ordering map with a permutation $\tau$. Then $\Theta(y) = I_k y$, where $I_k$ is the identity matrix $I_1$ with columns permuted according to $\tau$.

DEFINITION 6. *The set $S_i$ defined as*

$$S_i = \{y \in \mathbb{R}^m : (I_i y)_1 \geq (I_i y)_2 \geq \cdots \geq (I_i y)_m\},$$

*where $(I_i y)_j$ is the $j^{th}$ component of the vector $I_i y$, is called sector $i$.*

REMARK 3. Let $y \in \mathbb{R}^m$. Then $I_i y \in S_1$ iff $y \in S_i$.

PROPOSITION 2. *A sector is a convex cone.*

PROOF. Let $y \in S_i$ and $\alpha > 0$. We have that $(I_i y)_1 \geq (I_i y)_2 \geq \cdots \geq (I_i y)_m$ and also $I_i(\alpha y) = \alpha I_i(y)$, $i = 1, \dots, m$. We obtain $\alpha(I_i y)_1 \geq \alpha(I_i y)_2 \geq \cdots \geq \alpha(I_i y)_m$ so that $\alpha y \in S_i$, which proves that $S_i$ is a cone.

Let $y^1, y^2 \in S_i$. Clearly, $I_i y^1 + I_i y^2 = I_i(y^1 + y^2)$. Since $(I_i y^k)_j \geq (I_i y^k)_{j+1}$ for all $j = 1, \dots, m-1$ and $k = 1, 2$, we have $(I_i(y^1 + y^2))_j \geq (I_i(y^1 + y^2))_{j+1}$ for all $j = 1, \dots, m-1$. Thus $y^1 + y^2 \in S_i$ proving that $S_i$ is a convex cone. ∎

REMARK 4. A sector is not pointed.

PROPOSITION 3. *$\bigcap_{i=1}^{m!} S_i = l$, where $l = \{y \in \mathbb{R}^m : y_1 = y_2 = \cdots = y_m\}$ and is referred to as the equity line.*

PROOF. It is clear that $l \subset \bigcap_{i=1}^{m!} S_i$. Assume that $l \neq \bigcap_{i=1}^{m!} S_i$. Then, there exists a vector $y \in \mathbb{R}^m$ s.t. $y \in \bigcap_{i=1}^{m!} S_i$ with two different components $y_p \neq y_q$. Consider the matrix $I_s$ obtained by permuting columns $p$ and $q$ in $I_1$. Since $y \in \bigcap_{i=1}^{m!} S_i$ we have $y \in S_1$, or equivalently $y_1 \geq y_2 \geq \cdots \geq y_m$, and $y \in S_s$, or equivalently $y_1 \geq y_2 \geq \cdots y_{p-1} \geq y_q \geq y_{p+1} \geq \cdots \geq y_{q-1} \geq y_p \geq y_{q+1} \geq \cdots \geq y_m$, which yields $y_p = y_{p+1} = \cdots y_{q-1} = y_q$ contradicting the assumption. ∎

PROPOSITION 4. *For any two sectors, $S_i$ and $S_j$ in $\mathbb{R}^m$, their intersection defined as $S_{ij} := S_i \bigcap S_j = \{y \in \mathbb{R}^m : (I_i - I_j)y = \mathbf{0}\}$ is not empty.*

PROOF. Clearly, by Proposition 3, $S_{ij} \neq \emptyset$. Consider $y \in S_i \bigcap S_j$. Then by Remark 3, $I_i y \in S_1$ and $I_j y \in S_1$, which yields $I_i y = I_j y$. ∎

Let $\Delta_1$ denote the $m \times m$ lower triangular matrix of the form

$$\Delta_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

and $\Delta_k$, $k = 1, \ldots, m!$, denote the cumulative ordering matrices obtained by permuting columns of $\Delta_1$. Note that $\Delta_k = \Delta_1 I_k$.

REMARK 5. Let $y \in S_k$. Then $\bar{\Theta}(y) = \Delta_1 \Theta(y) = \Delta_1 I_k y = \Delta_k y$.

DEFINITION 7. *A permutation cone, $D_k \in \mathbb{R}^m$, is defined as a pointed convex cone of the form $D_k = \{d \in \mathbb{R}^m : \Delta_k d \geqq \mathbf{0}\}$, where $\Delta_k := \Delta_1 I_k$.*

PROPOSITION 5. *The following properties hold for permutation cones:*

1. $D_k = I_k^\top D_1$,
2. $D_s = I_s^\top I_k D_k$.

PROOF.

1. Let $d \in D_1$, then equivalently, $\Delta_1 I_1 d \geqq \mathbf{0}$ or $\Delta_1 d \geqq \mathbf{0}$. Consider the direction $I_k^\top d$ and calculate $\Delta_k(I_k^\top d) = \Delta_1 I_k I_k^\top d = \Delta_1 d \geqq \mathbf{0}$, which yields $I_k^\top d \in D_k$ and proves the result.
2. Similarly, let $d \in D_k$, then equivalently, $\Delta_k d \geqq \mathbf{0}$ or $\Delta_1 I_k d \geqq \mathbf{0}$. Consider the direction $I_s^\top I_k d$ and calculate $\Delta_s(I_s^\top I_k d) = \Delta_1 I_s I_s^\top I_k d = \Delta_1 I_k d = \Delta_k d \geqq \mathbf{0}$, which yields $I_s^\top I_k d \in D_s$ and proves the result. ∎

LEMMA 1. *A point $y \in \mathbb{R}^m$ is in a sector $S_k$ if and only if $\Delta_k y \geqq \Delta_p y$ for all $p = 1, \ldots, m!$.*

PROOF. Let $y \in S_k$ or by definition, $(I_k y)_1 \geq (I_k y)_2 \geq \cdots \geq (I_k y)_m$. Calculating the components of $\Delta_k y = \Delta_1 I_k y$ we obtain

$$\begin{aligned} (\Delta_k y)_1 = (\Delta_1 I_k y)_1 = (I_k y)_1 &\geq (I_p y)_1 = (\Delta_1 I_p y)_1 = (\Delta_p y)_1, \\ (\Delta_k y)_2 = (\Delta_1 I_k y)_2 = (I_k y)_1 + (I_k y)_2 &\geq (I_p y)_1 + (I_p y)_2 = (\Delta_1 I_p y)_2 = (\Delta_p y)_2, \\ &\vdots \\ (\Delta_k y)_m = \sum_{i=1}^m (I_k y)_i &\geq \sum_{i=1}^m (I_p y)_i = (\Delta_p y)_m, \end{aligned}$$

where the inequalities above result from the definition of $S_k$ and are equivalent to the desired vector inequality

$$\Delta_k y \geqq \Delta_p y, \qquad \text{for all } p = 1, \ldots, m!. \qquad ∎$$

REMARK 6. Let $y \in \mathbb{R}^m$. If $y \in S_k$ and $y \notin S_r$ then $\Delta_k y \geq \Delta_r y$.

COROLLARY 1. *Let $d \in D_k$. If there exists $y \in S_k$ such that $y + d \in S_p$, for some $p \in \{1, \ldots, m!\}$, then $d \in D_p$.*

PROOF. Let $y + d \in S_p$. Then by Lemma 1, $\Delta_p(y + d) \geqq \Delta_k(y + d)$ or $\Delta_p y + \Delta_p d \geqq \Delta_k y + \Delta_k d$. Since $d \in D_k$ we have $\Delta_k d \geqq \mathbf{0}$, and since $y \in S_k$ we have $\Delta_k y \geqq \Delta_p y$. The three last inequalities imply $\Delta_p d \geqq \Delta_k d \geqq \mathbf{0}$, which yields $d \in D_p$. ∎

## 4. PREFERENCE STRUCTURE OF EQUITABILITY

In this section, we relate the preliminary results with the concept of equitability. We first examine properties of two outcomes located in the same sector and in two different sectors.

PROPOSITION 6. *Let $y^1$ and $y^2$ be points in a sector $S_k$. If $y^1 \leqq y^2$ then $y^1 \succeq_e y^2$.*

PROOF. Let $y^1, y^2 \in S_k$ and $y^1 \leqq y^2$. Then also $I_k y^1 \leqq I_k y^2$, and by Remark 2, $\Theta(y^1) \leqq \Theta(y^2)$, which by Proposition 1 yields $\bar{\Theta}(y^1) \leqq \bar{\Theta}(y^2)$ and $y^1 \succeq_e y^2$. ∎

COROLLARY 2. *Let $y^k$ and $y^j$ be points in sectors $S_k$ and $S_j$, respectively, where $k \neq j$. If $I_k y^k \leqq I_j y^j$ then $y^k \succeq_e y^j$.*

PROPOSITION 7. *Let $y^1$ and $y^2$ be points in a sector $S_k$ and $D_k$ be the related permutation cone. Then $y^1 \succeq_e y^2$ iff $y^2 - y^1 \in D_k$.*

PROOF. Using Proposition 1 and Remark 5, we obtain

$$y^1 \succeq_e y^2 \overset{\text{Proposition 1}}{\Longleftrightarrow} \Delta_1 I_k y^1 \leqq \Delta_1 I_k y^2 \Leftrightarrow \Delta_1 I_k \left(y^2 - y^1\right) \geqq \mathbf{0} \Leftrightarrow y^2 - y^1 \in D_k. \qquad ∎$$

PROPOSITION 8. *Let $y^k$ and $y^j$ be points in sectors $S_k$ and $S_j$, respectively, where $k \neq j$. Then the following statements are equivalent:*

1. $y^k \succeq_e (\succ_e) y^j$,
2. $\Delta_k y^k \leqq (\leq) \Delta_j y^j$,
3. $I_j y^j - I_k y^k \in D_1(D_1 \setminus \{\mathbf{0}\})$,
4. $I_k y^k \succeq_e (\succ_e) I_j y^j$,
5. $y^j \in I_j^\top I_k(y^k + D_k)\ (I_j^\top I_k(y^k + D_k \setminus \{\mathbf{0}\}))$,
6. $y^k \in I_j^\top I_k(y^j - D_k)\ (I_j^\top I_k(y^j - (D_k \setminus \{\mathbf{0}\})))$.

PROOF. Since $y^k \in S_k$ and $y^j \in S_j$, by Proposition 1 and Remark 5 we obtain

$$y^k \succeq_e y^j \Leftrightarrow \Delta_1 I_k y^k \leqq \Delta_1 I_j y^j,$$

which is equivalent to (2) above and, by Definition 7, to (3) above. Additionally,

$$\Delta_1 I_k y^k \leqq \Delta_1 I_j y^j \Leftrightarrow \Delta_1 I_1 I_k y^k \leqq \Delta_1 I_1 I_j y^j.$$

Since $I_k y^k, I_j y^j \in S_1$ we obtain (4)

$$\Delta_1 \Theta\left(I_k y^k\right) \leqq \Delta_1 \Theta\left(I_j y^j\right) \Leftrightarrow \bar{\Theta}\left(I_k y^k\right) \leqq \bar{\Theta}\left(I_j y^j\right) \Leftrightarrow I_k y^k \succeq_e I_j y^j.$$

Also (3) is equivalent to

$$(3) \Leftrightarrow \exists\, d \in D_1 \text{ s.t. } I_j y^j = I_k y^k + d \Leftrightarrow y^j \in I_j^\top I_k y^k + I_j^\top D_1 \Leftrightarrow y^j \in I_j^\top I_k\left(y^k + D_k\right),$$

where the last equivalence results from Proposition 5. Similarly we obtain the equivalence with (6). ∎

PROPOSITION 9. *Let $S_k$ be a sector and $D_k$ the related permutation cone. If $y \in S_k$ and $d \in D_k$, $d \neq \mathbf{0}$ then $y + d \prec_e y$.*

PROOF. One of the following two cases may occur.

1. If $y + d \in S_k$, then by Proposition 7, $y + d \prec_e y$.
2. If $y + d \notin S_k$ then there exists a sector $S_p$, $p \neq k$ such that $y + d \in S_p$. Then by Lemma 1 we have $\Delta_p(y + d) \geqq \Delta_k(y + d)$. Since $d \in D_k$, we also have $\Delta_k d \geqq \mathbf{0}$. We obtain

$$\Delta_p(y + d) \geqq \Delta_k y + \Delta_k d \geq \Delta_k y,$$

and by Remark 5, $\bar{\Theta}(y + d) \geq \bar{\Theta}(y)$, which in turn gives $y + d \prec_e y$. ∎

COROLLARY 3. *Let be $y \in S_k$, $d \in D_k \setminus \{\mathbf{0}\}$, and $y + d \in S_p$, then $I_p(y + d) - I_k y \in D_1 \setminus \{\mathbf{0}\}$.*

PROOF. Immediate from Proposition 9. ∎

Note that the conclusion of Corollary 3 is equivalent to all other statements in Proposition 8.

Theorems 1 and 2 and Corollary 4 constitute the main results of this section. They provide a complete description of the equitability preference structure.

THEOREM 1. *Let be $\bar{y} \in S_k$. The set $D(\bar{y}) \in \mathbb{R}^m$ of points equitably dominated by $\bar{y}$ is given by*

$$D(\bar{y}) = \bigcup_{p=1}^{m!} I_p^\top I_k(\bar{y} + D_k \setminus \{\mathbf{0}\}).$$

PROOF. Recall that by definition, $D(\bar{y}) := \{y \in \mathbb{R}^m : y \prec_e \bar{y}\}$. First assume that $y \in D(\bar{y})$. Let $y \in S_p$ for some $p \in \{1, \ldots, m!\}$. Due to Proposition 8,

$$y \prec_e \bar{y} \Leftrightarrow y \in I_p^\top I_k(\bar{y} + D_k \setminus \{\mathbf{0}\}).$$

Since $p$ is arbitrary, we establish

$$\{y \in \mathbb{R}^m : y \prec_e \bar{y}\} \subseteq \bigcup_{p=1}^{m!} I_p^\top I_k\left(\bar{y} + D_k \setminus \{\mathbf{0}\}\right).$$

To complete the proof, assume that $y \in \bigcup_{p=1}^{m!} I_p^\top I_k(\bar{y} + D_k \setminus \{\mathbf{0}\})$. Then $y \in I_p^\top I_k(\bar{y} + D_k \setminus \{\mathbf{0}\})$ for some $p \in \{1, \ldots, m!\}$. Due to Proposition 5, we obtain $y \in I_p^\top I_k \bar{y} + D_p \setminus \{\mathbf{0}\}$ which implies that there exists $d \in D_p \setminus \{\mathbf{0}\}$ such that $y = I_p^\top I_k \bar{y} + d$. By Definition 7, we get $\Delta_p(y - I_p^\top I_k \bar{y}) \geq \mathbf{0}$, which becomes $\Delta_p y - \Delta_k \bar{y} \geq \mathbf{0}$. Then, by Proposition 8 we have $y \prec_e \bar{y}$. ∎

THEOREM 2. *Let be $\bar{y} \in S_k$. The set $P(\bar{y}) \in \mathbb{R}^m$ of points equitably preferred to $\bar{y}$ is given by*

$$P(\bar{y}) = \bigcap_{p=1}^{m!} I_p^\top I_k\left(\bar{y} - (D_k \setminus \{\mathbf{0}\})\right).$$

PROOF. Recall that by definition, $P(\bar{y}) = \{y \in \mathbb{R}^m : y \succ_e \bar{y}\}$. First assume that $y \in P(\bar{y})$. Let $y \in S_q$ for some $q \in \{1, \ldots, m!\}$.

Consider vectors defined as follows:

$$d_p := I_p^\top I_k \bar{y} - y, \qquad \forall\, p = 1, \ldots, m!,$$

and calculate

$$\begin{aligned} \Delta_p d_p &= \Delta_p I_p^\top I_k \bar{y} - \Delta_p y \\ &= \Delta_k \bar{y} - \Delta_p y \\ &\overset{\text{Lemma 1}}{\geqq} \Delta_k \bar{y} - \Delta_q y \\ &\overset{\text{Proposition 8}}{\geqq} \mathbf{0}, \end{aligned}$$

which yields $d_p \in D_p$ for all $p = 1, \ldots, m!$, where by Proposition 5, $D_p = I_p^\top I_k D_k$. Thus, for all $p = 1, \ldots, m!$ we have

$$y = I_p^\top I_k \bar{y} - d_p$$

or equivalently

$$y \in I_p^\top I_k\left(\bar{y} - D_k\right),$$

and since $d_p \neq 0$

$$y \in I_p^\top I_k \left(\bar{y} - D_k \setminus \{\mathbf{0}\}\right).$$

Therefore

$$y \in \bigcap_{p=1}^{m!} I_p^\top I_k \left(\bar{y} - D_k \setminus \{\mathbf{0}\}\right).$$

To complete the proof, assume that $y \in \bigcap_{p=1}^{m!} I_p^\top I_k(\bar{y} - (D_k \setminus \{\mathbf{0}\}))$. Then for every $p \in \{1, \dots, m!\}$ we have

$$\begin{aligned} y \in I_p^\top I_k \left(\bar{y} - (D_k \setminus \{\mathbf{0}\})\right) &\overset{\text{Proposition 5}}{\Longrightarrow} y \in I_p^\top I_k \bar{y} - D_p \setminus \{\mathbf{0}\} \\ &\Longrightarrow \exists\, d_p \in D_p \setminus \{\mathbf{0}\} \;\text{ s.t. } y = I_p^\top I_k \bar{y} - d_p \\ &\Longrightarrow d_p = I_p^\top I_k \bar{y} - y \in D_p \setminus \{\mathbf{0}\}. \end{aligned}$$

By Definition 7 we obtain

$$\Delta_p \left(I_p^\top I_k \bar{y} - y\right) \geq \mathbf{0},$$

which using Remarks 5 and 1 becomes

$$\Delta_k \bar{y} - \Delta_p y \geq \mathbf{0}$$

and by Proposition 8 yields

$$y \succ_e \bar{y}.$$

∎

PROPOSITION 10. *$P(\bar{y})$ is a convex set.*

PROOF. Let $\dot{y}$, $\ddot{y} \in P(\bar{y})$ and assume that $\bar{y} \in S_k$, $\dot{y} \in S_p$, and $\ddot{y} \in S_q$ where $k, p, q \in \{1, \dots, m!\}$. We show that $\alpha\dot{y} + (1-\alpha)\ddot{y} \in P(\bar{y})$ for all $\alpha \in (0,1)$.

Assume that for an arbitrary $\alpha \in (0,1)$, $\alpha\dot{y} + (1-\alpha)\ddot{y} \in S_r$ for some $r \in \{1, \dots, m!\}$. Calculate

$$\begin{aligned} \Delta_r(\alpha\dot{y} + (1-\alpha)\ddot{y}) &= \Delta_r\alpha\dot{y} + \Delta_r(1-\alpha)\ddot{y} \\ &= \alpha\Delta_r\dot{y} + (1-\alpha)\Delta_r\ddot{y} \\ &\overset{\text{Lemma 1}}{\leqq} \alpha\Delta_p\dot{y} + (1-\alpha)\Delta_q\ddot{y}. \end{aligned}$$

Since $\dot{y} \succ_e \bar{y}$ and $\ddot{y} \succ_e \bar{y}$, by Proposition 8 we get $\Delta_p\dot{y} \leq \Delta_k\bar{y}$ and $\Delta_q\ddot{y} \leq \Delta_k\bar{y}$, respectively. Then we continue

$$\begin{aligned} \alpha\Delta_p\dot{y} + (1-\alpha)\Delta_q\ddot{y} &\leq \alpha\Delta_k\bar{y} + (1-\alpha)\Delta_k\bar{y} \\ &= \Delta_k\bar{y}. \end{aligned}$$

In summary, $\Delta_r(\alpha\dot{y} + (1-\alpha)\ddot{y}) \leq \Delta_k\bar{y}$ which, by Proposition 8, is equivalent to $\alpha\dot{y} + (1-\alpha)\ddot{y} \in P(\bar{y})$. ∎

COROLLARY 4. *Let be $\bar{y} \in S_k$. The set $I(\bar{y}) \in \mathbb{R}^m$ of points equitably indifferent to $\bar{y}$ is given by*

$$I(\bar{y}) = \left\{y \in \mathbb{R}^m : y = I_p^\top I_k \bar{y} \text{ for some } p = 1, \dots, m!\right\}.$$

We end this section with some results on outcomes obtained through the application of the principle of transfers.

DEFINITION 8. *Let $y^1$ and $y^2$ be in $\mathbb{R}^m$. We say that $y^2$ has been obtained from $y^1$ using the principle of transfers a finite number of times if there exist a finite sequence of vectors $y^{10} = y^1, y^{11}, \dots, y^{1(t-1)}$, $y^{1t} = y^2$ such that $y^{1k} = y^{1k-1} - \varepsilon_k e_{i'} + \varepsilon_k e_{i''}$, $0 < \varepsilon_k < y_{i'}^{k-1} - y_{i''}^{k-1}$ for $k = 1, 2, \dots, t$.*

Let $PT(\bar{y})$ denote the set of all points generated using the principle of transfers a finite number of times starting with $\bar{y}$. The following proposition will be used in the next section.

PROPOSITION 11. *Let $y^1$ and $y^2$ be in $\mathbb{R}^m$. If $y^1 \succeq_e y^2$ and $\bar{\theta}_m(y^1) = \bar{\theta}_m(y^2)$ then $y^1$ can be obtained from $y^2$ using the principle of transfers a finite number of times.*

PROOF. Without loss of generality and due to impartiality, assume that $y^1, y^2 \in S_1$, i.e., $y_1^1 \geq y_2^1 \geq \cdots \geq y_m^1$ and $y_1^2 \geq y_2^2 \geq \cdots \geq y_m^2$. If $y^1 \succeq_e y^2$ then $y_1^1 \leq y_1^2$. Using the principle of transfers for each component $y_k^2$ of $y^2$ such that $y_k^2 > y_1^1$, and applying $\sum_{i=1}^m y_i^1 = \sum_{i=1}^m y_i^2$, we can find $\bar{y}^2$ for which

$$\max_{i=1,\dots,m} \bar{y}_i^2 = y_1^1.$$

Without loss of generality assume now that $\bar{y}^2 \in S_1$. Then using the principle of transfers we obtain

$$y^1 \succeq_e \bar{y}^2, \tag{1}$$

$$y_1^1 = \bar{y}_1^2, \tag{2}$$

$$\sum_{i=1}^m y_i^1 = \sum_{i=1}^m \bar{y}_i^2, \tag{3}$$

$$y_2^1 \leq \bar{y}_2^2. \tag{4}$$

Due to (1)–(4), again using the principle of transfers for each component $\bar{y}_k^2$ of $\bar{y}^2$, where $\bar{y}_k^2 > y_2^1$ and $k \geq 2$, we can get $\hat{y}^2 \in S_1$ such that

$$\begin{aligned} y^1 &\succeq_e \hat{y}^2, \\ y_1^1 &= \hat{y}_1^2, \\ y_2^1 &= \hat{y}_2^2, \\ \sum_{i=1}^m y_i^1 &= \sum_{i=1}^m \hat{y}_i^2, \\ y_3^1 &\leq \hat{y}_3^2. \end{aligned}$$

Note that first two components of $y^1$ and $\hat{y}^2$ are the same. Repeating the same procedure a finite number of times we construct $\tilde{y}^2$ such that $\tilde{y}^2 = y^1$. ∎

Based on Definitions 2 and 8 we also obtain the following result.

PROPOSITION 12. *Let $\bar{y} \in \mathbb{R}^m$. Then $PT(\bar{y}) \subset P(\bar{y})$.*

## 5. GENERATING EQUITABLY EFFICIENT SOLUTIONS

Consider the following multiobjective program (MOP):

$$\begin{aligned} \min \quad & f(x) = [f_1(x), f_2(x), \dots, f_m(x)], \\ \text{s.t.} \quad & x \in X \subseteq \mathbb{R}^n, \end{aligned} \tag{5}$$

where $f$ is a vector-valued function and $f_i$, $i = 1, \dots, m$, are real-valued functions. Consider two types of efficiency for this MOP, Pareto efficiency and equitable efficiency. We say that a feasible solution $x$ is Pareto efficient for the MOP if there does not exist $x' \in X$ such that $f(x') \leq f(x)$. We say that a feasible solution $x$ is equitably efficient for the MOP if there does not exist $x' \in X$ such that $f(x') \succ_e f(x)$. The outcome $y = f(x)$ of a (Pareto or equitably) efficient solution $x$ is referred to as (Pareto or equitably) nondominated.

Theorem 3 generalizes the analogous result developed for linear MOPs in [9].

THEOREM 3. *Let $x \in X$ and $y = f(x)$. If $y$ is an equitably nondominated outcome for the MOP then it is also Pareto nondominated for this problem.*

PROOF. Recall that $y - \mathbb{R}^m_{\geqq}$ is the set of points preferred to $y$ with respect to Pareto efficiency. Note that $\mathbb{R}^m_{\geqq} \subset D_p$, $\forall p = 1, \ldots, m!$. Hence, $y - \mathbb{R}^m_{\geqq} \subset P(y)$. Thus, the set of all equitably nondominated outcomes is contained in the set of all Pareto nondominated outcomes. ∎

Theorem 3 motivates the development of a two-step method for finding equitably efficient solutions in the sense that a suitably found Pareto efficient solution of the MOP may lead to an equitably efficient solution. Consider first the following $\varepsilon$-constrained scalarization of the MOP in which the objective functions are added but also individually bounded from above by $\varepsilon$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} f_i(x), \\ \text{s.t.} \quad & f_i(x) \le \varepsilon, \qquad \text{for all } i = 1, \ldots, m, \\ & x \in X \subseteq \mathbb{R}^n. \end{aligned} \tag{6}$$

An optimal solution of problem (6) is a Pareto efficient solution of the MOP [13]. Let $z_\varepsilon$ denote the optimal objective value of problem (6) and consider another single-objective program in which the Euclidean norm of the outcome vector is minimized subject to the feasibility and optimality of problem (6).

$$\begin{aligned} \min \quad & \left\| [f_1(x), f_2(x), \ldots, f_m(x)] \right\|, \\ \text{s.t.} \quad & \sum_{i=1}^{m} f_i(x) = z_\varepsilon, \\ & f_i(x) \le \varepsilon, \qquad \text{for all } i = 1, \ldots, k, \\ & x \in X \subseteq \mathbb{R}^n. \end{aligned} \tag{7}$$

THEOREM 4. *Let $x^* \in X$. If $x^*$ is an optimal solution for problem (7) then it is an equitably efficient solution for the MOP (5).*

PROOF. Let $x^*$ be an optimal solution for (7). By contradiction assume that $x^*$ is not an equitably efficient solution for the MOP (5), i.e., there exists $\bar{x} \in X$ such that $f(\bar{x}) = \bar{y} \succ_e y^* = f(x^*)$. Using Proposition 1 we have

$$\bar{y} \succ_e y^* \iff \bar{\Theta}(\bar{y}) \le \bar{\Theta}(y^*).$$

From the first component of this vector inequality we get

$$\max_{i=1,\ldots,m} \bar{y}_i \le \max_{i=1,\ldots,m} y_i^* \le \varepsilon,$$

where the second inequality above holds since $x^*$ is feasible for problem (7). From the last component of the same vector inequality we obtain

$$\sum_{i=1}^{m} f_i(\bar{x}) \le \sum_{i=1}^{m} f_i(x^*) = z_\varepsilon.$$

However, since $z_\varepsilon$ is the optimal value of problem (6), there must be $\sum_{i=1}^{m} f_i(\bar{x}) = z_\varepsilon$, which makes $\bar{y}$ feasible for problem (7), and also $\bar{\theta}_m(\bar{y}) = \bar{\theta}_m(y^*)$. Using now Proposition 11, we obtain that $\bar{y}$ can be obtained from $y^*$ using the principle of transfers a finite number of times. Therefore $\|\bar{y}\| < \|y^*\|$, which contradicts the assumption that $x^*$ is an optimal solution for problem (7). ∎

COROLLARY 5. *If $x^*$ is an optimal solution for problem (6) with $y^* = f(x^*)$ such that $y_1^* = y_2^* = \cdots = y_m^*$, then it is an equitable efficient solution to the MOP (5).*

PROOF. Under the given assumptions, $x^*$ is also an optimal solution for problem (7), and by Theorem 4, is equitably efficient for the MOP. ∎

Problems (6) and (7) constitute a two-step method for finding equitably efficient solutions of the MOP. The Pareto nondominated outcome produced by problem (6) leads to an equitably nondominated outcome of the MOP. Note that problems (6) and (7) are single objective nonlinear programs that can be solved using conventional optimization methods.

## 6. CONCLUSION

While equitability has not been given much attention in the mathematical programming literature, with this article, its theory and methodology have been advanced. The complete preference structure of equitability is derived and a scalarization approach to finding equitably efficient solutions of general multiple objective programs is proposed.

Further studies may go in the direction of applied research, in particular, applications of equitability in engineering design.

## REFERENCES

1. P.L. Yu, Cone convexity, cone extreme points and nondominated solutions in decision problems with multiobjectives, *Journal of Optimization Theory and Applications* **14**, 319–377, (1974).
2. J. Figueira, S. Greco and M. Ehrgott, *Multiple Criteria Decision Analysis: State of the Art Surveys*, Springer, New York, (2005).
3. K. Bergstresser, A. Charnes and P.L. Yu, Generalization of domination structures and nondominated solutions in multicriteria decision making, *Journal of Optimization Theory and Applications* **18** (1), 3–13, (1976).
4. E. Takeda and T. Nishida, Multiple criteria decision problems with fuzzy domination structures, *Fuzzy Sets and Systems* **3**, 123–136, (1980).
5. G.B. Hazen and T.L. Morin, Optimality conditions in nonconical multiple-objective programming, *Journal of Optimization Theory and Applications* **40** (1), 25–60, (1983).
6. P. Weidner, Problems in scalarizing multicriteria approaches, In *Multiple Criteria Decision Making in the New Millennium*, (Edited by S. Zionts and M. Köksalan), pp. 199–209, Springer, Berlin, (2001).
7. P. Weidner, Tradeoff directions and dominance sets, In *Multi-Objective Programming and Goal Programming*, (Edited by T. Tanaka *et al.*), pp. 275–280, Springer, Berlin, (2003).
8. G.Y. Chen and X.Q. Yang, Characterization of variable domination structures via nonlinear scalarization, *Journal of Optimization Theory and Applications* **112** (1), 97–110, (2002).
9. M.M. Kostreva and W. Ogryczak, Linear optimization with multiple equitable criteria, *RAIRO Operations Research* **33** (3), 275–297, (1999).
10. M.M. Kostreva, W. Ogryczak and A. Wierzbicki, Equitable aggregations in multiple criteria analysis, *European Journal of Operational Research* **158** (2), 362–377, (2004).
11. W. Ogryczak, Multiple criteria linear programming model for portfolio selection, *Annals of Operations Research* **97**, 143–162, (2000).
12. W. Ogryczak, Inequality measures and equitable approaches to location problems, *European Journal of Operational Research* **122** (2), 374–391, (2000).
13. R.E. Wendell and D.N. Lee, Efficiency in multiple objective optimization problems, *Mathematical Programming* **12** (3), 406–414, (1977).